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AN ANALYTIC OBSTRUCTION TO A COMPLEX TORAL ACTION ON A COMPLEX MANIFOLD

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Introduction

A theorem of R. Bott asserts that the Chern numbers of a compact complex manifold M can be computed from suitable invariants defined at the zeros of any nondegenerate holomorphic vector field on M. In particular, if M admits a nowhere vanishing holomorphic vector field, its Chern numbers vanish. Since one may view such a vector field as a holomorphic action of the complex numbers on M, we see that *if a compact complex manifold M admits a holomorphic action of a complex Lie group without fixed points, the Chern numbers of M must vanish.* For example, this is always the case if the group is compact. The purpose of the present note is to show that with this restriction on the action other cohomology classes must vanish too.

We will say a holomorphic vector-field X is *invariant* if it preserves a Hermitian structure g on M, i.e., if $L_X g = 0$. Note that we are assuming X is of type $(1,0)$ so $X = Y - iJY$ where Y and JY are both infinitesimal isometries and J is the complex structure of M. Such a vector field arises in particular when M admits a holomorphic action of a complex torus. An invariant vector field is nowhere zero if either it arises from a torus action as above or the metric is Kaehlerian. The refined Chern classes of a complex manifold are certain cohomology classes in the refined de Rham cohomology ring of M which are defined from a Hermitian structure on M, but actually independent of the choice of structure; they generate a graded ring denoted $\widehat{ch}(M)$. By Hodge theory, the refined cohomology ring of a compact Kaehler manifold agrees with the usual de Rham cohomology ring.

In view of the above result of Bott, it is natural to expect that the existence of an invariant vector field on M has implications for the refined Chern ring. We state a result in this direction.

Theorem 1. *Suppose M is a Hermitian manifold admitting invariant vector fields $X_1, \cdots, X_k$ of type $(1,0)$ which are linearly independent at some point $p \in M$. Then the refined Chern ring of M vanishes in dimensions exceeding $2(\dim_C M - k)$.*

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A holomorphic foliation on M is a holomorphic subbundle E' of $T'M$ which is a foliation; i.e., whenever $X, Y \in \underline{E}'$, the module of C^∞ sections of E', then $[X, Y] \in \underline{E}$. Let $F' = T'M/E'$ be the quotient bundle with holomorphic projection $p\colon T'M \to F'$. There exists a linear operator

$$\underline{E}' \otimes \underline{F}' \to \underline{F}'$$

called the *natural E'-connection on F'* in [1] defined by

$$\nabla_X s = p[X, \tilde{s}]$$

for $X \in \underline{E}'$ and $s \in \underline{F}'$, where $\tilde{s}$ is any lift of s to $\underline{T'M}$. This operator satisfies
(i) $\nabla_{fX}s = f\nabla_X s$, (ii) $\nabla_X fs = (Xf)s + f\nabla_X s$,
for any $f \in C^\infty(M)$, and hence the operator

$$\nabla_{X_p}\colon \underline{F}' \to F'_p$$

is well defined for $X_p \in E'_p$. Let us say that the holomorphic foliation E' is *Hermitian* if there exists a Hermitian form $\langle\ ,\ \rangle$ on F' with

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle$$

for all $X \in E'_p$ and all $s, t \in \underline{F}_p$, $\underline{F}$ being the sheaf of germs of holomorphic sections of F'.

Then we can prove

Theorem 2. *If E' is a Hermitian foliation on M of dimension k, then the refined Chern ring of F' vanishes in dimensions exceeding* $2(\dim_C M - k) = 2\dim F'$.

Note that we are not assuming M is compact here. Theorem 2 yields a vanishing criterion for certain almost free group actions including complex toral (holomorphic) actions. On a compact Kaehler manifold M, Theorems 1 and 2 are related in the following way: the complex vector space of parallel vector fields on M corresponds precisely (under the natural complex map $TM \to T'M$) to the space of invariant vector fields. Hence the invariant vector fields generate a Hermitian foliation on M.

In the Riemannian case, a group G of isometries of a compact manifold M acting almost freely on M induces a metric on $F = TM/E$ (E being the natural foliation) such that for all $p \in M$

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$$

for all $X \in E_p$ and $s, t \in \underline{F}$. In [6] Joel Pasternack proved that this implies that the real Pontryagin ring of M vanishes in dimensions exceeding $(\dim_R M - \dim_R G)$.

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1. A theorem on foliations

Let us begin by proving the result of Bott which says that if M admits a holomorphic foliation E', then the Chern ring of $T'M/E'$ vanishes in dimensions exceeding 2 dim (TM'/E').

A connection on E' is a differential operator $D\colon \underline{E}' \to \underline{T^*M} \otimes \underline{E}'$ satisfying (i) $D(s_1 + s_2) = Ds_1 + Ds_2$, (ii) $D(fs) = df \otimes s + fDs$, for all $s, s_1, s_2 \in \underline{E}'$ and $f \in C_C^\infty(M)$. With respect to any local holomorphic frame $\underline{s} = \{s_1, \cdots, s_k\}$ for E', D is given by a matrix $\theta(\underline{s}; D)$ of 1-forms defined by

$$Ds_i = \sum \theta_{ij} \otimes s_j \ .$$

We say that D is of type $(1, 0)$ if each θ_{ij} is a form of type $(1, 0)$. The curvature matrix of D with respect to $\underline{s}$ is

$$K(\underline{s}; D) = d\theta(\underline{s}; D) - \theta(\underline{s}; D) \wedge \theta(\underline{s}; D) \ .$$

We say that a symmetric polynomial on $gl(k, C)$ is ad-invariant if

$$\phi(X_1, \cdots, X_m) = \phi(AX_1A^{-1}, \cdots, AX_mA^{-1})$$

for any $A \in GL(k, C)$. Because of the way the curvature transforms under a change of frame, for any invariant polynomial ϕ,

$$\phi(K) = \phi(K(\underline{s}; D), \cdots, K(\underline{s}, D))$$

is a global form on M, and it turns out that $\phi(K)$ is always closed. The cohomology class of $\phi(K)$ is an element of the Chern ring of M.

Now since E' is a foliation, one may always choose coordinates $z_1, \cdots, z_n$ on M such that $\partial/\partial z_1, \cdots, \partial/\partial z_k$ span E' on, say, U_α. Define a connection ∇^α for F' on U_α by

$$i(\partial/\partial z_l)\nabla^\alpha p(\partial/\partial z_j) = 0$$

for $l = 1, \cdots, n$ and $j = k + 1, \cdots, n$. One checks that ∇^α is of type $(1, 0)$ and that $\nabla^\alpha_W Y = p[W, \tilde{Y}]$ for any $W \in \underline{E}' | U_\alpha$ and $Y \in \underline{F}' | U_\alpha$; that is, ∇^α restricts to the natural E'-connection on F' over U_α. Thus, if $\{\phi_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$, then the connection

$$\nabla = \sum \phi_\alpha \nabla^\alpha$$

is of type $(1, 0)$ and is an extension of the natural E'-connection on F'. By a similar construction, when M has a nowhere vanishing holomorphic vector field X, it is possible to define a connection D on M of type $(1, 0)$ such that

$$i(X)DV = [X, V]$$

for any vector field V on M.

With respect to the frame $\underset{\sim}{s} = \{p\,\partial/\partial z_{k+1}, \cdots, p\,\partial/\partial z_n\}$ for F' over U, we therefore have

$$i(W)\theta(\underset{\sim}{s}; \bar{V}) = 0$$

for any $W \in \underline{E' \mid U}$. Thus for any homogeneous invariant polynomial ϕ of degree exceeding $(n - k)$, $\phi(K(\underset{\sim}{s}; \bar{V}))$ vanishes due to the fact that θ is of type $(1, 0)$, and this completes the proof.

2. The refined Chern ring

If M is any complex manifold and E' any Hermitian vector bundle on M, there exists on E' a unique Hermitian connection D of type $(1, 0)$. Thus

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle$$

for any $s_1, s_2 \in \underline{E'}$, where $\langle\ ,\ \rangle$ denotes the Hermitian form on E'. If $\underset{\sim}{s} = \{s_1, \cdots, s_n\}$ is a local holomorphic frame on E', and $N = \|\langle s_i, s_j\rangle\|$, then one can show that

$$\theta(\underset{\sim}{s}; D) = d'N\cdot N^{-1}\ , \qquad K(\underset{\sim}{s}, D) = d''\theta(\underset{\sim}{s}; D)\,;$$

that is, the curvature is of type $(1, 1)$.

Let $\hat{H}^k(M) = A^{k,k} \cap \ker d/id'd''A^{k-1,k-1}$, $A^{k,k}$ being the module of C^∞ forms on M of type (k, k). The $\hat{H}^k(M)$ define a graded ring $H^*(M) = \sum \hat{H}^k(M)$. If one considers a Hermitian vector bundle E' with associated curvature $K(E')$, each homogeneous invariant polynomial ϕ gives rise to an element $\hat{\phi}(E')$ of the refined Chern ring $\widehat{ch}(E')$ of E'. The k-th refined Chern class $\hat{c}_k(E')$ of E' is defined by the terms of type (k, k) in $\det(I + (1/2\pi i)K(E'))$. We set $\hat{c} = 1 + \sum \hat{c}_j$. Then: (i) the cohomology class $\hat{\phi}(E')$ is independent of the Hermitian structure on E'; (ii) for any exact sequence of holomorphic vector bundles over M

$$0 \to H' \to E' \to F' \to 0\ ,$$

there is the relation $\hat{c}(E') = \hat{c}(H')\hat{c}(F')$; and (iii) if E' is holomorphically trivial, then $\hat{c}(E') = 1$. The proofs of (i)-(iii) are found in [4].

3. The vanishing theorem

When M has a nonvanishing holomorphic vector field X, we have seen that M has a good connection, i.e., one for which $i(X)\theta(\underline{s}; D) = 0$ for certain frames $\underline{s}$. In fact, if M has Hermitian metric g, and the vector field X preserves the metric, in other words, $L_X g = 0$, then we will show that the canonical Hermitian connection D is good, at least on a dense subset of M. Let $\hat{M} = M - \text{zero}\,(X)$. We will prove that, for any invariant polynomial ϕ,

$$i(X)\phi(K) = 0 \qquad \text{on } \hat{M} ,$$

where K denotes the curvature of D. Since $\hat{M}$ is dense in M, $i(X)\phi(K) = 0$ on M.

Let $\underline{s} = \{s_1, \cdots, s_n\}$ be a local coordinate frame on $\hat{M}$ with $s_1 = X$, and let $N = \|\langle s_i, s_j\rangle\|$. Then recalling the definition of D,

$$i(X)\theta(\underline{s}; D) = i(X)(d'N\cdot N^{-1}) = (i(X)d'N)\cdot N^{-1} = (L_X N)\cdot N^{-1} = 0 ,$$

since by assumption $L_X N = 0$. Thus $i(X)\phi(K(\underline{s}; D)) = 0$. But $\phi(K)$ is independent of the frame and hence it vanishes along any invariant vector field on M, which proves Theorem 1.

Unfortunately Theorem 1 does not follow directly from Theorem 2 since $X_1, \cdots, X_k$ need not span a foliation. Suppose they do. We note that the Hermitian metric on M extends to a bilinear complex tensor on $TM \otimes_R C$ denoted by $(\,,)$. The Hermitian form on $T'M$ is then

$$\langle X, Y\rangle = 2(X, \bar{Y}) ,$$

where bar denotes conjugation. Thus, if X is invariant and s, t are local holomorphic sections of $T'M$, then

$$X\langle s, t\rangle = 2(L_X g)(s, \bar{t}) + 2([X, s], \bar{t}) + 2(s, [X, \bar{t}]) ,$$

and this reduces to $X\langle s, t\rangle = \langle [X, s], t\rangle$, so E' is Hermitian.

Example 1. If M is a compact Kaehler manifold with complex structure J, then the correspondence $Y \to Y - iJY$ gives a bijection between the set of parallel vector fields on M and the set of invariant vectors fields on M. In fact, if Y is parallel with respect to the Kaehler structure, so is JY, and $Y - iJY$ is invariant since a parallel vector field is an infinitesimal isometry. Conversely, suppose that $X = Y - iJY$ is an invariant vector field on any Kaehler manifold. Now on a Kaehler manifold the Hermitian connection D of type $(1, 0)$ coincides with the Kaehlerian (i.e., Riemannian) connection. But we have already observed that

$$i(X)DV \doteq [X, V]$$

for any complex vector field V on M, and

$$i(X)DV - i(V)DX = [X, V]$$

since D is torsion free. Thus $DX = 0$, so Y and JY are indeed parallel since D is a real operator.

As a corollary we note that *an invariant vector field on a Kaehler manifold never vanishes.*

Example 2. Suppose W is homogeneous Kaehler manifold written as G/H, where G is connected, has nondiscrete center, and preserves the Kaehler structure, and suppose H is compact. Lichnerowicz [5] proved that one can always find complex coordinates near each point of W so that the metric can be written

$$ds^2 = dz_1 d\bar{z}_1 + \sum_{\alpha,\beta \geq 2} g_{\alpha\beta} dz_\alpha d\bar{z}_\beta$$

with $g_{\alpha\beta}$ independent of z_1 and $\bar{z}_1$. From this it follows that $\widehat{ch}(W)$ vanishes in the top dimension. If one considers a one parameter subgroup σ of the center of G such that $\sigma \not\subset H$, the vector field X on W of type $(1, 0)$ preserves the metric. Since G acts as holomorphic transformations of W, X is holomorphic and hence invariant.

Example 3. Suppose G is a complex torus acting holomorphically on a complex manifold M (i.e., the map $G \times M \to M$ is holomorphic). By the usual averaging process one may assume M has a Hermitian structure invariant under G. Now given a complex flow $(z, m) \to \phi_z(m)$ on $M(z \in C)$ induced by G, one defines a holomorphic vector field X on M by

$$X_m = \frac{d}{dz}\phi_z(m)|_{z=0} .$$

The real flows ϕ_s and $\phi_{it}(s, t \in R)$ induce infinitesimal isometries Y and Z of M satisfying $JY = Z$ since the action is holomorphic. Hence a complex flow induced on M by a complex torus gives an invariant vector field. Let us say that G acts with rank k at $m \in M$ if the complex rank of the map

$$v \to \frac{d}{dz} \exp(zv)m|_{z=0}$$

of $T_eG \to T'_mM$ is k. From Theorem 1 we have the following: *A complex toral action of rank k (at some point) on a complex manifold M ensures the vanishing of the refined Chern ring of M in the top $2k$ dimensions.*

Let us now prove Theorem 2. Let $\tilde{\nabla}$ denote the canonical Hermitian connection on F' of type $(1, 0)$. We will prove that $\tilde{\nabla}$ is an extension of the natural E'-connection on F'. As before choose complex coordinates $z_1, \cdots, z_n$ for M so that $\partial/\partial z_1, \cdots, \partial/\partial z_k$ span $\underline{E}'$ locally and set $s_j = p\,\partial/\partial z_{k+j}$. Then $\underline{s} = \{s_{k+1}, \cdots, s_n\}$ is a local frame for F', and clearly $\nabla_W s_j = 0$ if $W \in \underline{E}'$. Let $N = \|\langle s_i, s_j\rangle\|$. Then

$$i(W)\theta(\underline{s};\tilde{V}) = i(W)d'N\cdot N^{-1} = (WN)\cdot N^{-1} = \|\langle V_W s_i, s_j\rangle\|\cdot N^{-1} = 0\ ,$$

and thus $\theta(\underline{s};\tilde{V})$ vanishes along E'. Hence $\tilde{V}$ extends the natural E'-connection, and the assertion follows.

Example. The foliation coming from a group action in the case of a complex torus of dimension k acting holomorphically and effectively is holomorphically trivial, so the refined Chern classes $\hat{c}_i(M)$ vanish for $i > n - k$. The corollary asserts that the Chern ring vanishes for these dimensions too. On the other hand, every even dimensional compact connected Lie group has a left invariant complex structure, but its holomorphic tangent bundle is certainly not holomorphically trivial in general since a compact complex parallelizable manifold is the quotient of a complex Lie group by a discrete subgroup. However, we may always suppose that the maximal torus of G sits inside G as a complex torus. By considering the action of T on G on the right we see that $\widehat{ch}(G)$ vanishes in dimensions exceeding ($\dim_R G$ − rank G). Note that the left action of T on G is never holomorphic.

4. A vanishing theorem for group actions

Suppose G is a compact Lie group acting almost freely on a smooth manifold M. Thus the differential of $g \to gm$ at the identity e of G is of maximal rank for each $m \in M$. From this one obtains a smooth (trivial) foliation E of M as the image of the natural injection

$$M \times T_eG \to TM\ .$$

Since G is compact, we may suppose that there is a metric on M invariant under G. Then a one-parameter subgroup of G induces a Killing vector field X on M so that

$$X(Y, Z) = ([X, Y], Z) + (Y, [X, Z])$$

for all $Y, Z \in \underline{TM}$. Let p: $TM \to F$, $F = TM/E$, be the canonical map. If we suppose $p|E^{\perp}$ is an isometry, then the induced fibre metric on F is compatible with the natural E-connection on F, i.e.,

$$(1) \qquad X(s, t) = (V_X s, t) + (s, V_X t)$$

for all $X \in \underline{E}$, and $s, t \in \underline{F}$.

Now suppose in addition that M is a complex manifold having complex structure J and that G is a group of holomorphic transformations of M leaving invariant a Hermitian metric on M. Let q: $TM \to T'M$ be the natural identification mapping $X \to X' = \frac{1}{2}(X - iJX)$. If $E' = q(E)$ and $F' = T'M/E'$, *then* we have a diagram of homomorphisms:

$$\begin{array}{ccc} TM & \xrightarrow{q} & T'M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ F & \xrightarrow{q'} & F' \end{array}$$

If we assume that $J(E) = E$, then we may induce a Hermitian fibre metric on F and a Hermitian form on F' in the usual way, and these are related in the usual may:

$$\langle s', t' \rangle = (s, t) + i(s, Jt)$$

for $s, t \in \underline{F}$ since $q(E^{\perp}) = E'^{\perp}$ in this case.

Lemma. *Suppose the foliation E on M is J-invariant (i.e., $J(E) = E$). Then the natural metric on F' is invariant in the sense that for all $p \in M$,*

$$X\langle s', t' \rangle = \langle \nabla_{X'} s', t' \rangle$$

for all $X' \in E'_p$ and $s', t' \in F'_p$.

Proof. First note that

$$\nabla_{X'} s' = \tfrac{1}{4}\{\nabla_X s - \nabla_{JX} Js - i(\nabla_{JX} s + \nabla_X Js)\} .$$

Thus

$$2\langle \nabla_{X'} s', t' \rangle = (\nabla_X s, t) - (\nabla_{JX} Js, t) + i(\nabla_X s, Jt) - i(\nabla_{JX} Js, Jt) ,$$

and similarly

$$2\langle s', \nabla_{X'} t' \rangle = (s, \nabla_X t) - (s, \nabla_{JX} Jt) - i(s, \nabla_{JX} t) + i(s, \nabla_X Jt) .$$

Now since s and t are holomorphic, $\nabla_{JX} s = J\nabla_X s$, etc., so applying the compatibility condition (1) for F, we see that

$$X\langle s', t' \rangle = \langle \nabla_{X'} s', t' \rangle + \langle s', \nabla_{X'} t' \rangle .$$

Moreover, since $(JX)' = iX'$,

$$iJX\langle s', t' \rangle = -\langle \nabla_{X'} s', t' \rangle + \langle s', \nabla_{X'} t' \rangle .$$

Thus $X'\langle s', t' \rangle = \langle \nabla_{X'} s', t' \rangle$.

Theorem 4. *Suppose G is a compact Lie group of holomorphic transformations acting almost freely on a complex manifold M, and the (real) foliation E induced by G on M is J-invariant. Then $\widehat{ch}(T'/E)$ vanishes in dimensions exceeding $2(\dim_C M - \frac{1}{2}\dim_R G)$ provided E is holomorphic.*

Corollary. *Suppose G is a complex torus acting effectively and holomorphically on a complex manifold M. Then the action is almost free, and hence the conclusion of Theorem 4 holds for $\widehat{ch}(M)$.*

Proof. To show that the action of G is almost free we must prove that $G_m = \{g \in G: gm = m\}$ is finite for each $m \in M$. Suppose at m_0 this is not so. Then G_{m_0} is a closed, hence compact, complex subgroup of G leaving m_0 fixed. We will prove that G_{m_0} acts as the identity on M. Indeed, consider the linear isotropy representation $g \to (d'g)_{m_0}$ of G_{m_0} in $GL(T'M_{m_0})$. This representation is holomorphic and hence trivial on the identity component K of G_{m_0} by the maximum principle. But it is well known that any compact connected Lie group action can be locally imbedded in a linear action near a fixed point. By the triviality of the linear isotropy representation, K acts locally as the identity in a neighborhood of m_0, i.e., there are a neighborhood U of e in K and a neighborhood V of m_0 in M such that the map $U \times V \to M$ has image m_0. But again using the analyticity, this is impossible unless $K = \{e\}$, contradicting our assumption on G_{m_0}.

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